

Analysis of Failure Loads of Timber Bridges



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Abstract: A method has been presented for finding the expected failure capacity of a bridge, expressed as a "reduction factor" times the failure capacity obtained by taking all beams to have the same MOR and E_L , equal to the mean values for the species of wood concerned. The reduction factor depends upon the transverse deflection profile and is at its smallest for a uniform transverse deflection. For Red Pine, the "expectation" or expected value of the reduction factor is 0.609 for the case of uniform deflection. For design purposes, a reduction factor corresponding to 99.9% confidence is used. This reduction factor depends upon the number of girders in the bridge but is close to the bottom end of the frequency distribution. For Red Pine, a reduction factor of 0.532 is recommended.			
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1/ INTRODUCTION

There are two properties of timber beams which have a marked effect on the distribution of live loads in timber laminated bridges and on the ultimate load carrying capacity of such bridges. These are the modulus of elasticity and the modulus of rupture.

The modulus of elasticity (E_L) and modulus of rupture (MOR) vary quite widely from one timber beam to another. Figure 1 shows test values of E_L and MOR on seventy Red Pine specimens. These values were originally given in an unpublished report from the Ontario Ministry of Transportation and Communications, and were used recently in a paper by Bakht and Jaeger¹. The variability in E_L and MOR has the effect of significantly reducing the ultimate load carrying capacity of the bridge, as compared to what it would be if all beams had the same E_L and MOR, equal to their mean values for the species concerned. This significant reduction can be related to the amount of scatter that is shown in Figure 1. The reduction factor is also dependent upon the deflection profile of a cross-section of the bridge.

2/ DERIVATION OF THE METHOD OF ANALYSIS

As a first step, the values of E_L and MOR given in Figure 1 are replotted on axes of $\left[\frac{E_L}{(E_L)_{\text{mean}}} \right]$ and $\left[\frac{\text{MOR}}{(\text{MOR})_{\text{mean}}} \right]$ as shown in Figure 2. Then, for a given species of timber, the scatter of points can be approximated as lying within an area bounded by $r = r_1$ and $r = r_2$, and by $\theta = \alpha$ and $\theta = 90^\circ - \alpha$, where r is a radial distance from the origin and θ is an angle measured clockwise from the $\left[\frac{E_L}{(E_L)_{\text{mean}}} \right]$ axis, as is shown in Figure 2. For Red Pine, the angle α is about 30° . It will be shown later that the limiting values r_1 and r_2 do not need to be estimated. The scatter of points over the area of Figure 2 is represented by a probability density function $p(r, \theta)$ which satisfies the relationship:

$$\int_{\alpha}^{\frac{\pi}{2} - \alpha} \int_{r_1}^{r_2} p(r, \theta) r dr d\theta = 1 \quad (1)$$

For simplicity, two further assumptions are made about the nature of the scatter of points in Figure 2. These are that the probability distribution can be split up into "variation with θ " and "variation with r " and that the "variation with θ " is symmetrical about the line $\theta = 45^\circ$. Both of these assumptions are reasonable in light of the observed test values for different species of timber. The general nature of the probability distribution $p(r, \theta)$ is as shown in Figure 3a. Then we require:

$$p(r, \theta) = f(r) \phi(\theta) \quad (2)$$

where:

$$\int_{r_1}^{r_2} f(r) r dr = 1 \quad (3)$$

and:

$$\int_{\alpha}^{\frac{\pi}{2} - \alpha} \phi(\theta) d\theta = 1 \quad (4)$$

Figure 3b shows the general nature of the probability function $\phi(\theta)$. This function must be zero outside the range of values α and $90^\circ - \alpha$ and

be symmetrical about $\theta = 45^\circ$. A suitable form for the purposes of analysis is:

$$\phi(\theta) = c(\sin 2\theta - \sin 2\alpha) \quad (5)$$

for:

$$\alpha < \theta < 90^\circ - \alpha$$

The constant c in Equation 5 is readily found by using Equation 4 to be $[\cos 2\alpha - (\frac{\pi}{2} - 2\alpha) \sin 2\alpha]^{-1}$. In the case of red pine, taking $\alpha = 30^\circ$, one then obtains:

$$\phi(\theta) = 21.488(\sin 2\theta - \frac{\sqrt{3}}{2}). \quad (5a)$$

Having derived the necessary properties of the scatter of value pairs of E_L and MOR, an investigation of ultimate failure loads is now undertaken for various patterns of transverse deflection. For each pattern of deflection, the actual expected value of ultimate failure load is related to the deterministic failure load which is obtained by taking all beams to have the same E_L and MOR, being the mean value for the timber species concerned. As a preliminary it may be noted that the ultimate failure load, M_{ult} , of a beam is directly proportional to its MOR. Hence

in Figure 2 the horizontal axis can be $\left[\frac{M_{ult}}{(M_{ult})_{mean}} \right]$ as well as $\left[\frac{MOR}{(MOR)_{mean}} \right]$.

3/ THE CASE OF UNIFORM DEFLECTION

The first pattern of transverse deflection that is examined is that of uniform deflection, shown in Figure 4. If all beams are taken to be identical, then as the uniform deflection increases and the bending moments accepted by the beam increase, all beams will simultaneously reach $(M_{ult})_{mean}$. Hence, in the deterministic approach, the total ultimate bending moment capacity of the bridge is simply:

$$M_T = N(M_{ult})_{mean} \quad (6)$$

N is the number of beams

When the scatter of values of E_L and MOR is taken into account, a very different behaviour emerges. The beams do not all fail simultaneously. Since all beams deflect equally, the moment accepted by a beam is proportional to its own E_L and its ability to withstand moment is proportional to its own value of MOR, (or, alternatively, its own value of M_{ult}). The first beams to fail are hence those with the lowest value of

MOR. Figure 5 is a repeat of Figure 2 with the addition of a failure E_L boundary at $\theta = \beta$. As the uniform deflection shown in Figure 4 increases, a radial line starting at the vertical axis rotates clockwise, as shown in Figure 5. For a certain deflection, all of those beams which are to the left of the failure boundary in Figure 5 have already failed, whilst those to the right have not. Similarly to Csagoly and Taylor² we assume that when a beam's ultimate moment capacity has been reached the beam fails, and its moment carrying capacity then falls to zero.

The total bending moment being accepted by those beams that have not yet failed is readily found. We consider the small element $r\delta r\delta\theta$ shown at the point S in Figure 5a. The number of beams expected in this area is $Np(r,\theta)r\delta r\delta\theta$. These beams are not yet at their failure moments. They are in fact taking a fraction $\frac{NP}{NS}$ of their failure moments, with the notation of Figure 5a. This follows because all beams having the same value of E_L take the same moments as one another, until failure. Using the notation shown in Figure 5a we have:

$$NP = r\cos\theta\tan\beta \quad (7)$$

The bending moment accepted by the beams in the shaded area $r\delta r\delta\theta$ is:

$$\delta M = \{Np(r,\theta)r\delta r\delta\theta\}r\cos\theta\tan\beta(M_{ult})_{mean} \quad (8)$$

Using Equations 2, 3, 4 and 6 and by integrating Equation 8 we then obtain:

$$M = M_T \tan\beta \int_{r_1}^{r_2} f(r)r^2 dr \int_{\beta}^{\frac{\pi}{2} - \alpha} \cos\theta\phi(\theta)d\theta \quad (9a)$$

which is conveniently written:

$$M = k(\beta)M_T \quad (9b)$$

where:

$$k(\beta) = \tan\beta \int_{r_1}^{r_2} f(r)r^2 dr \int_{\beta}^{\frac{\pi}{2} - \alpha} \cos\theta\phi(\theta)d\theta \quad (9c)$$

The calculation of $k(\beta)$ using Equation 9c is straightforward and is greatly simplified since the function $f(r)$ does not need to be found explicitly. In fact, the value of $\int_{r_1}^{r_2} f(r)r^2 dr$ can be determined from knowing the position of the centre of gravity of the probability distribution. For details about the evaluation of Equation 9c refer to Figure 5b and consider the moment of the probability distribution $p(r,\theta)$ about the line QQ shown. The centre of gravity of this probability distribution is at the point G whose co-ordinates are (1,1). Hence the distance OG is $\sqrt{2}$. Since the total of the probability distribution (i.e., $\iint p(r,\theta)rdrd\theta$) is equal to one, we have:

Moment of probability distribution around QQ:

$$M_{QQ} = 1 \times \sqrt{2} = \sqrt{2} \quad (10)$$

Now considering this moment by integration of elements, and again putting $p(r,\theta) = f(r)\phi(\theta)$ we take the typical element shown, which is a perpendicular distance $r\sin(\frac{\pi}{4} + \theta)$ from the line QQ. The contribution of the

probability at this element to moment around QQ is:

$$\delta M_{QQ} = f(r)\phi(\theta) r \delta r \delta \theta \cdot r \sin\left(\frac{\pi}{4} + \theta\right) \quad (11)$$

whence, integrating for all elements:

$$M_{QQ} = \int_{\alpha}^{\frac{\pi}{2} - \alpha} \sin\left(\frac{\pi}{4} + \theta\right) \phi(\theta) d\theta \int_{r_1}^{r_2} f(r) r^2 dr \quad (12)$$

By Equations 10 and 12 we thus find that:

$$\int_{\alpha}^{\frac{\pi}{2} - \alpha} \sin\left(\frac{\pi}{4} + \theta\right) \phi(\theta) d\theta \int_{r_1}^{r_2} f(r) r^2 dr = \sqrt{2} \quad (13)$$

Eliminating $\int_{r_1}^{r_2} f(r) r^2 dr$ between Equations 9c and 13, and noting that $\sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{\sqrt{2}} (\sin\theta + \cos\theta)$ we obtain the relatively simple expression for $k(\beta)$ as follows:

$$k(\beta) = 2 \tan \beta \frac{\int_{\beta}^{\left(\frac{\pi}{2} - \alpha\right)} \cos \theta \cdot \phi(\theta) d\theta}{\int_{\alpha}^{\left(\frac{\pi}{2} - \alpha\right)} (\sin \theta + \cos \theta) \phi(\theta) d\theta} \quad (14)$$

The value of $k(\beta)$ is readily evaluated for any probability function $\phi(\theta)$. On substituting the value of $\phi(\theta)$ given by Equation 5 one obtains, on cancelling the constant c top and bottom,

$$k(\beta) = 2 \tan \beta \frac{\int_{\beta}^{\left(\frac{\pi}{2} - \alpha\right)} \cos \theta (\sin 2\theta - \sin 2\alpha) d\theta}{\int_{\alpha}^{\left(\frac{\pi}{2} - \alpha\right)} (\sin \theta + \cos \theta) (\sin 2\theta - \sin 2\alpha) d\theta} \quad (15)$$

For Red Pine, putting $\alpha = 30^\circ$, the equation for $k(\beta)$ is given as:

$$k(\beta) = 30.588 \tan \beta \left(\frac{2}{3} \cos^3 \beta + \frac{\sqrt{3}}{2} \sin \beta - \frac{5}{6} \right) \quad (16)$$

for values of β greater than 30° .

Although the details of the mathematics may not be of much interest to engineers, the underlying physics of how the situation builds up to the point of collapse certainly is. Referring again to Figures 4 and 5a, for small values of deflection the boundary $\theta = \beta$ does not go through the area defined by the scatter of points. As deflection continues to increase, however, and the radial line continues its motion, a stage is reached at which the radial line reaches the angular position α of Figure 5a. From that point on the most vulnerable beams (i.e., those with the smallest values of $\frac{MOR}{E_L}$) begin to fail. For a time, increasing

values of deflection and the corresponding increases in the angle β are accompanied by further increases in the load carrying capacity, but the rate of increase reduces steadily. Eventually a value of β is reached for which $k(\beta)$ is a maximum, after which a decrease occurs. The maximum value of $k(\beta)$ gives the failure state of the bridge in accordance with Equation 6.

Figure 6 is a graph of ultimate load factor $k(\beta)$ for Red Pine plotted against the angle β . The maximum is reached when β is 0.62 radians, i.e., 35.52° , at which point $k(\beta)$ has the value 0.639. Hence, for Red Pine in uniform deflection:

$$\begin{aligned} M &= 0.639 M_T \\ &= N \{0.639 (M_{ult})_{mean}\} \end{aligned} \quad (17)$$

Hence, from Equation 17, the ultimate failure load of the bridge is reduced, because of the scatter of values, to 0.639 of what it would have been if all beams had had the same M_{ult} , i.e., if all had had $(M_{ult})_{mean}$ and the same value of E_L , i.e., $(E_L)_{mean}$.

It should be emphasized that the reduction in the ultimate failure load as estimated by Equation 17 is not brought about by having beams which are, on the average, weaker than usual. The beams are taken to have an average strength which is entirely normal, and it is the scatter of $\frac{MOR}{E_L}$ above and below the mean which gives rise to the progressive failure of the bridge and the reduction in its ultimate load carrying capacity as compared with the "no scatter" situation.

The reduction factor of 0.639 which appears in Equation 17 is derived from the probability distribution of the quantity $\frac{MOR}{E_L}$, and the ultimate failure moment M given by Equation 17 is in fact the mean value (i.e., the "expectation") of a random variable. The derivation of a failure load for design purposes also requires the determination of the standard deviation of this random variable. That determination is given later in this report. It may be noted immediately that for timber laminated bridges the number, N , of beams is large and hence the variability of the ultimate failure load about the mean value given by Equation 17 is small.

It is instructive at this point to examine how sensitive the value of $k(\beta)$ is to the probability function of $\phi(\theta)$ that is assumed. If, as an extreme case, we assume $\phi(\theta)$ to be constant for values of θ between α and $(\frac{\pi}{2} - \alpha)$ then we have:

$$k(\beta) = \frac{\tan\beta(\cos\alpha - \sin\beta)}{(\cos\alpha - \sin\alpha)} \quad (14a)$$

For Red Pine with $\alpha = 30^\circ$, this has the value of 0.577 when $\beta = 30^\circ$ and then decreases. We conclude that the value of $k(\beta)$ is not very sensitive to the assumed form of $\phi(\theta)$.

4/ THE CASE OF LINEARLY VARYING DEFLECTION

The adoption of a completely uniform deflection pattern across a transverse cross-section is, of course, a highly idealized case, which can be regarded as one extreme. For purposes of comparison, the second deflection pattern considered is that of Figure 7, in which the deflection is assumed to vary in a straight line from zero at one side of the bridge to a maximum value at the other. If all beams are taken to be identical with modulus of elasticity $(E_L)_{\text{mean}}$ and ultimate moment $(M_{\text{ult}})_{\text{mean}}$ then failure is reached when the deflection of the right hand end reaches the value which gives $(M_{\text{ult}})_{\text{mean}}$ at that end.

Making again the assumption that the moment of resistance of a failed beam falls immediately to zero, it is readily seen that attainment of $(M_{\text{ult}})_{\text{mean}}$ at the right end leads at once to collapse of the bridge and that:

$$M_T = 0.5N (M_{\text{ult}})_{\text{mean}} \quad (18)$$

In Equation 18, the factor 0.5 arises from the linear variation of deflection across the cross-section.

The estimation of total failure moment for this deflection pattern, taking into account the scatter in values of $\frac{\text{MOR}}{E_L}$, follows the same general

line as that for the uniform deflection case, except that now each element of the cross-section has its own deflection and hence its own radial line on the plot of Figure 2. Figures 8a and 8b show the situation in which the deflection of the right end gives the radial line $\theta = \beta$. Then a width $B \frac{\tan \alpha}{\tan \beta}$ of the cross-section is known to be still unfailed, whilst the right hand portion of the cross-section is composed of elements which have increasing probabilities of failure as one moves from left to right.

The contribution to moment of resistance given by the left hand width $B \frac{\tan \alpha}{\tan \beta}$, say $N(M_{\text{ult}})_{\text{mean}} k_1(\beta)$ is:

$$N(M_{ult})_{mean} k_1(\beta)$$

$$= \int_0^{\frac{B \tan \alpha}{\tan \beta}} \frac{N}{B} (M_{ult})_{mean} \frac{y}{B} \tan \beta dy \int_{\alpha}^{\frac{\pi}{2} - \alpha} \cos \theta \phi(\theta) d\theta \int_{r_1}^{r_2} f(r) r^2 dr \quad (19)$$

whence:

$$k_1(\beta) = \int_0^{\frac{B \tan \alpha}{\tan \beta}} \frac{y}{B^2} \tan \beta dy \int_{\alpha}^{\frac{\pi}{2} - \alpha} \cos \theta \phi(\theta) d\theta \int_{r_1}^{r_2} f(r) r^2 dr \quad (20)$$

Once again using Equation 13 to eliminate the $\int_{r_1}^{r_2} f(r) r^2 dr$, we have:

$$k_1(\beta) = \frac{\frac{B \tan \alpha}{\tan \beta}}{2} \int_0^{\frac{B \tan \alpha}{\tan \beta}} \frac{y}{B^2} \tan \beta dy \frac{\int_{\alpha}^{\frac{\pi}{2} - \alpha} \cos \theta \phi(\theta) d\theta}{\int_{\alpha}^{\frac{\pi}{2} - \alpha} (\sin \theta + \cos \theta) \phi(\theta) d\theta} \quad (21a)$$

$$\text{whence } k_1(\beta) = \frac{\tan^2 \alpha}{2 \tan \beta} = \frac{1}{6 \tan \beta} \text{ when } \alpha = 30^\circ$$

The contribution to moment of resistance given by the part of the cross-section from $\frac{B \tan \alpha}{\tan \beta}$ to B, say $N(M_{ult})_{mean} k_2(\beta)$, is found in a very similar way, except that each element of this part of the beam has its own radial line (say $\theta = \gamma$) which goes through the probability zone. We have:

$$\tan \gamma = \frac{y}{B} \tan \beta \quad (21b)$$

and:

$$k_2(\beta) = \int_{\frac{B \tan \alpha}{\tan \beta}}^B y \frac{\tan \beta}{B^2} \left[\int_{\gamma}^{\frac{\pi}{2} - \alpha} \cos \theta \phi(\theta) d\theta \int_{r_1}^{r_2} f(r) r^2 dr \right] dy \quad (22)$$

On eliminating $\int_{r_1}^{r_2} f(r) r^2 dr$ we have:

$$k_2(\beta) = \int_B^B \frac{\tan \alpha}{\tan \beta} \left[\frac{y \tan \beta}{B} \frac{\int_{\gamma}^{\left(\frac{\pi}{2} - \alpha\right)} \cos \theta \phi(\theta) d\theta}{\int_{\alpha}^{\left(\frac{\pi}{2} - \alpha\right)} (\sin \theta + \cos \theta) \phi(\theta) d\theta} \right] dy \quad (22a)$$

Since, as noted above in Equation 21b the angle γ is itself a function of y , the integration shown in square brackets in Equation 22a is performed first. The integrals are all straightforward, and they give as the result:

$$k_2(\beta) = \frac{30.588}{\tan \beta} \times \left[.665420 - \frac{2}{3} \cos \beta - \frac{5}{12} \tan^2 \beta + \frac{\sqrt{3}}{4} \tan \beta \sec \beta - \frac{\sqrt{3}}{4} \log(\tan \beta + \sec \beta) \right] \quad (22b)$$

From equations 21 and 22b we have:

$$k(\beta) = k_1(\beta) + k_2(\beta) = \frac{30.588}{\tan \beta} \times \left[.670873 - \frac{2}{3} \cos \beta - \frac{5}{12} \tan^2 \beta + \frac{\sqrt{3}}{4} \tan \beta \sec \beta - \frac{\sqrt{3}}{4} \log(\tan \beta + \sec \beta) \right] \quad (23)$$

Tabulating $k(\beta)$ against β then gives:

$\beta(\text{radians})$	0.60	0.65	0.70	0.75	0.80	0.805	0.81	0.85
$\beta(\text{degrees})$	34.38	37.24	40.11	42.97	45.84	46.12	46.41	48.70
$k(\beta)$	0.340	0.370	0.393	0.409	0.414	0.415	0.414	0.410

The maximum is seen to occur at 0.805 radians. Hence we have:

$$M = 0.415 N (M_{ult})_{\text{mean}} \quad (23a)$$

In this case it is seen from Equations 12 and 13 that the reduction factor in ultimate load carrying capacity is 0.830, as compared to 0.639 in the case of a uniform deflection. This relatively good performance is entirely to be expected since in the pattern of deflection which is now assumed, a fairly large fraction of the cross-section is known to be not involved in the initiation of failure. Therefore, the influence of the scatter of values of $\frac{MOR}{E_L}$ is much reduced.

The procedure for estimating ultimate moment capacity for any general deflection pattern, such as that shown in Figure 9, follows the same steps. The maximum deflection of the cross-section corresponds to a radial line $\theta = \beta$ in the plot of Figure 5. Any other point on the cross-section has a radial line $\theta = \gamma$ where $\frac{\tan \gamma}{\tan \beta}$ is the ratio of deflections for the two points concerned.

In practice, the reduction factor for a uniform deflection, which is 0.639 in the case of Red Pine, is found to be the lowest reduction factor that occurs, as can be seen from the following analysis.

5/ THE CASE OF DEFLECTION VARYING LINEARLY BETWEEN TWO NON-ZERO VALUES

To complete the work on estimation of ultimate failure loads, a third pattern of deflection is now considered. This is shown in Figure 10. The deflection varies linearly from a minimum value corresponding to the radial line $\theta = \beta_0$ in Figure 10a to a maximum value corresponding to $\theta = \beta$.

The portion AB of the cross section is of length $B \left\{ \frac{\tan \alpha - \tan \beta_0}{\tan \beta - \tan \beta_0} \right\}$ and consists of beams that are still undamaged. At a distance y from the left hand end, in this part of the cross section, the deflection corresponds to a radial line whose tangent is:

$$\tan \beta_0 + \frac{y}{B} (\tan \beta - \tan \beta_0) = \frac{y \tan \beta + (B - y) \tan \beta_0}{B}$$

and with the notation introduced earlier:

$$M_T k_1(\beta) =$$

$$\int_0^{B \left\{ \frac{\tan \alpha - \tan \beta_0}{\tan \beta - \tan \beta_0} \right\}} \left\{ \frac{N \delta y}{B} \right\} (M_{ult})_{mean} \left\{ \frac{y \tan \beta + (B - y) \tan \beta_0}{B} \right\} dy \quad (24)$$

$$\int_{\alpha}^{\pi - \alpha} \frac{1}{2} \cos \phi(\theta) d\theta \int_{\sqrt{I}}^{\sqrt{I}} f(r) r^2 dr$$

For Red Pine:

$$k_1(\beta) = \frac{1.0008}{B} \int_0^{B \left\{ \frac{\tan \alpha - \tan \beta_0}{\tan \beta - \tan \beta_0} \right\}} \left[\tan \beta_0 + \frac{y}{B} (\tan \beta - \tan \beta_0) \right] dy \quad (25)$$

$$= 1.0008 \left\{ \frac{\tan \beta_0 (\tan \alpha - \tan \beta_0)}{(\tan \beta - \tan \beta_0)} + \frac{1}{2} \frac{(\tan \alpha - \tan \beta_0)^2}{(\tan \beta - \tan \beta_0)} \right\}$$

$$= \frac{1.0008}{2} \left\{ \frac{(\tan \alpha - \tan \beta_0)(\tan \alpha + \tan \beta_0)}{(\tan \beta - \tan \beta_0)} \right\} = 0.5004 \left[\frac{\frac{1}{3} - \tan^2 \beta_0}{\tan \beta - \tan \beta_0} \right]$$

For a typical loading (see below) $\tan \beta_0 = 0.1783$, in which case:

$$k_1(\beta) = \frac{0.15089}{\tan\beta - 0.1783} \quad (26)$$

Similarly, for Red Pine:

$$k_2(\beta) = 30.588 \int_B^B \left\{ \frac{\tan\alpha - \tan\beta_0}{\tan\beta - \tan\beta_0} \right\} \frac{y \tan\beta + (B - y) \tan\beta_0}{B^2} dy \int_Y^{\frac{\pi}{3}} \cos\theta \left\{ \sin 2\theta - \frac{\sqrt{3}}{2} \right\} d\theta$$

$$\text{when } \tan\gamma = \tan\beta_0 + \frac{y}{B} (\tan\beta - \tan\beta_0)$$

$$\text{Hence } k_2(\beta) = 30.588 \int_{\alpha = \frac{\pi}{6}}^{\beta} \frac{1}{B} \tan\gamma \frac{B \sec^2\gamma}{(\tan\beta - \tan\beta_0)} \left\{ \frac{2}{3} \cos^3\gamma + \frac{\sqrt{3}}{2} \sin\gamma - \frac{5}{6} \right\} d\gamma \quad (27)$$

$$\text{This gives } k_2(\beta) = \frac{30.588}{(\tan\beta - \tan\beta_0)} \left\{ .665420 - \frac{2}{3} \cos\beta - \frac{5}{12} \tan^2\beta + \frac{\sqrt{3}}{4} \tan\beta \sec\beta - \frac{\sqrt{3}}{4} \log(\sec\beta + \tan\beta) \right\} \quad (28)$$

It may be noted that Equations 25 and 27 reduce to 21 and 26 respectively on putting $\beta_0 = 0$.

Adding Equations 26 and 28 gives $k(\beta)$ which may be tabulated against β as follows:

$\beta(\text{radians})$	0.60	0.65	0.70	0.75	0.77	0.80
$\beta(\text{degrees})$	34.38	37.24	40.11	42.97	44.11	45.84
$k(\beta)$	0.428	0.455	0.475	0.484	0.485	0.483

Hence we have:

$$M = 0.485 N(M_{ult})_{\text{mean}} \quad (29)$$

For the deflection pattern assumed, the deflection varies linearly from one proportional to $\tan\beta_0$ (i.e., 0.1783) to $\tan\beta$ (i.e., 0.9697) and the average deflection is 0.592 of the maximum deflection. The deterministic approach would thus give an ultimate failure moment of $0.592 N(M_{ult})_{mean}$ and the reduction factor on this given by Equation 29 is 0.819.

In the three different deflection profiles that have been considered so far, a trend of behaviour becomes apparent as can be seen in the following table.

Maximum Deflection Average Deflection	Failure Moment Taking All Beams Equal	Failure Moment Taking Scatter	R
1.0	$N(M_{ult})_{mean}$	$0.639 N(M_{ult})_{mean}$	0.639
1.689	$0.592 N(M_{ult})_{mean}$	$0.485 N(M_{ult})_{mean}$	0.819
2.0	$0.5 N(M_{ult})_{mean}$	$0.415 N(M_{ult})_{mean}$	0.830

The reduction factor R is given by dividing the entry in column III by the entry in column II; i.e., it is the ratio of failing moments with and without scatter for various values of maximum deflection/average deflection. Repeating the calculations for other values of deflection, the graph of Figure 11 is obtained.

Recommended Procedure for Estimating the Ultimate Failure Condition

1. Treat all beams as equal, having $(M_{ult})_{mean}$;
2. On that basis do a deterministic analysis and find the ultimate carrying capacity and the deflection pattern;
3. Use Figure 11 to find the reduction factor that must be applied to the ultimate capacity found in Equation 2 to allow for scatter.

6/ ESTIMATION OF THE NUMBER OF BEAMS THAT HAVE FAILED WHEN ULTIMATE LOAD IS REACHED

For the case of uniform deflection, the estimated number of beams to fail is:

$$N \int_{\alpha}^{\beta} \phi(\theta) d\theta = N \int_{\alpha=30^{\circ}}^{\beta=35.523^{\circ}} 21.488(\sin 2\theta - \sin 60^{\circ}) d\theta, \text{ for Red Pine} \quad (30)$$

$$= (8.85 \times 10^{-2})N$$

For the case of deflection varying linearly from zero at one side, the estimate of the number of beams that have failed is:

$$\int_B^B \frac{\tan \alpha}{\tan \beta} \frac{N}{B} \left[\int_{\alpha}^{\gamma} \phi(\theta) d\theta \right] dy =$$

$$\frac{N}{B} (21.488) \int_{\frac{B \tan \alpha}{\tan \beta}}^B \left\{ \frac{1}{4} - \frac{\cos 2\gamma}{2} - \frac{\sqrt{3}}{2} \left(\gamma - \frac{\pi}{6} \right) \right\} dy$$

for Red Pine, where $\tan \gamma = \frac{y}{B} \tan \beta$ (31)

$$= \frac{(21.488)N}{1.04} \int_{\frac{\pi}{6}}^{0.805} \left(1.20345 \sec^2 \gamma - 1 - \frac{\sqrt{3}}{2} \gamma \sec^2 \gamma \right) d\gamma$$

$$= (10.40 \times 10^{-2})N$$

For the case of deflection varying linearly between two non-zero values the estimate of the number of beams that have failed is

$$\int_B^B \left\{ \frac{\tan \alpha - \tan \beta_0}{\tan \beta - \tan \beta_0} \right\} \frac{N}{B} \left[\int_{\alpha}^{\gamma} \phi(\theta) d\theta \right] dy$$

For Red Pine $= \frac{N}{B} (21.488) \int_B^B \left\{ \frac{\tan \alpha - \tan \beta_0}{\tan \beta - \tan \beta_0} \right\} \left\{ \frac{1}{4} - \frac{\cos 2\gamma}{2} - \frac{\sqrt{3}}{2} \left(\gamma - \frac{\pi}{6} \right) \right\} dy,$

where $\tan \gamma = \tan \beta_0 + \frac{y}{B} (\tan \beta - \tan \beta_0)$ (32)

$$= \frac{(21.488)N}{(\tan\beta - \tan\beta_0)} \int_{\alpha = \frac{\pi}{6}}^{\beta = 0.77} (1.20345 \sec^2\gamma - 1 - \frac{\sqrt{3}}{2} \gamma \sec\gamma^2) d\gamma = 0.0913N$$

It is noted that the estimated number of beams that fail prior to ultimate load is not very sensitive to the transverse deflection pattern, being about one tenth of the total number of beams in the bridge in most cases.

7/ A REVISED ESTIMATE OF THE ANGLE α FOR RED PINE

Since failure occurs when only a small percentage of the beams have failed, it follows that in order to be as accurate as possible in our estimate of failure load we should choose a value of α which represents the lower end of the probability distribution very closely and that preciseness of fit at the higher end of the distribution is much less important. With this in mind, a detailed investigation of the smaller values of $\left\{ \frac{MOR}{E_L} \right\} / \left\{ \frac{(MOR)_{mean}}{(E_L)_{mean}} \right\}$ was

undertaken from the seventy experimental values available. This showed an isolated, lowest value of 0.433, with next lowest values of 0.562, 0.566, 0.584,.... . This behaviour is much better represented by $\alpha = 28^\circ$, for which the lowest value of the variable would be 0.532, than by $\alpha = 30^\circ$, for which the lowest value would be 0.577. Accordingly $\alpha = 28^\circ$ is preferred.

For this value of α , Equation 16 is replaced by:

$$k(\beta) = 21.220 \tan \beta \left\{ \frac{2}{3} \cos^3 \beta + 0.82904 \sin \beta - 0.80098 \right\} \quad (24)$$

and the maximum value of $k(\beta)$ is found to be 0.609, obtained when $\beta = 34.7^\circ$.

8/ ESTIMATE OF THE VARIANCE OF THE FAILURE LOAD

We return now to the uniform deflection case, for which the expected value of the reduction factor is $k = 0.639$ if $\alpha = 30^\circ$, and 0.609 if $\alpha = 28^\circ$ and seek to estimate the variance of the failure load. It rapidly emerges that the behaviour of the random variable concerned is very far from a normal distribution so that it is a better approach to seek a failure load with a suitably low probability of occurrence, rather than to use the standard deviation of the failure load, per se. To do this, a semi-numerical technique is useful.

In establishing the expected value of failure load, a two dimensional plot of probability involving γ and θ was used; see, for example, Figures 2 and onward. In the estimation of the expected value of failure load, the distribution with respect to r , contained in the calculation as $\int_{r_1}^{r_2} f(r)r^2 dr$, turned out to be irrelevant. This is not entirely an accident. The representation of probability density as a function of r and θ is a very convenient one, and in particular is useful for suggesting an analytical technique with a symmetric formulation. If, however, purely numerical methods are turned to, we can replace this two dimensional representation of probability by a one dimensional (line) one, with only a very small loss of accuracy.

Figure 12a shows, in histogram form, the distribution of the non-dimensional quantity $x = (MOR/E) / \left\{ \frac{(MOR)_{mean}}{(E_L)_{mean}} \right\}$. With one isolated exception, this quantity has values which vary from about 0.56 to 1.59 for Red Pine, using the seventy values that are plotted in Figure 1. Figure 11b shows the same distribution when smoothed into a continuous function. In fact, the quantity x in Figure 11 corresponds to $\tan\theta$ in the plot of Figure 2, so that rotation of the radial line in Figure 2 is the same thing as advancement from left to right along the abscissa line of Figure 11, with the two motions being related by $x = \tan\theta$. It is noted that Figure 11a is closely similar to Figure 4 of Reference [1], being changed simply by a normalization of the horizontal co-ordinate with respect to $\left\{ \frac{MOR_{mean}}{E_L_{mean}} \right\}$.

If we now consider a gradually increasing uniform deflection of the bridge cross-section, represented by a gradual advance of the point P from left to right in the plot of Figure 12b, we consider the state of affairs when the uniform deflection corresponds to failure of beams whose $\left\{ \frac{MOR}{E_L} \right\}$ values are smaller than x . With the notation of Figure 11, and recalling that M_{ult} is proportional to MOR_{ult} , the total bending moment being sustained is closely given by:

$$M = N(M_{ult})_{mean} xg(x)$$

so that:

$$k = xg(x) \quad (26)$$

provided that the mean E_L of the unfailed beams in $g(x)$ is still the same as $(E_L)_{mean}$ for the entire set.

The behaviour of k as x increases from zero is as follows:

- (i) for values of x smaller than OA, $g(x)$ is equal to 1.0 and so $k = x$;
- (ii) as the moving point P passes across A, x continues to increase, but $g(x)$ now starts to diminish;
- (iii) for k to be a maximum, we require $g + x \frac{dg}{dx} = 0$, and it is clear that $\frac{dg}{dx} = -h$, where h is the height of the probability density at the location concerned.

Hence, the required maximum value of k is obtained when P reaches the point at which $OP \times PN = \text{area under the probability curve to the right of PN}$.

8.1/ Graphical Representation of k

Figure 13 shows the graph of $g(x)$ and the graph of $xg(x)$ for the particular distribution given by the seventy values for Red Pine. The maximum value of k is found to be 0.614, which agrees fairly closely with that obtained with $\alpha = 30^\circ$, and very closely indeed if the value of α is revised from 30° to 28° .

8.2/ Estimation of a Lower Bound Value of k for Design Purposes

We now seek a value of k which, to a chosen high level of confidence, will be a safe lower bound for design purposes. We recall that when dealing with normal Gaussian distributions, a level of two standard deviations below the mean gives a 97.7% probability of safety, while three standard deviations, increases the level to 99.9%. We therefore choose the latter level of confidence and seek the value of k that corresponds to it.

The expected fraction of total number of beams that have failed when a bridge collapses is:

$$\begin{aligned} \int_{\alpha}^{\beta} \phi(\theta) d\theta &= \int_{\alpha}^{\beta} c(\sin 2\theta - \sin 2\alpha) d\theta \\ &= c \left[\frac{1}{2} (\cos 2\alpha - \cos 2\beta) - (\beta - \alpha) \sin 2\alpha \right] \end{aligned} \quad (27)$$

For Red Pine with $\alpha = 30^\circ$, this fraction is about 8.8%, whilst with $\alpha = 28^\circ$, it is 10.0%. These estimates agree excellently with observed results [2].

Adopting a failure fraction of 10% as a basis of comparison, we now seek to find what minimum value of β will cause failure with a probability of not more than 1 in 1000, and calculate k from Equation 24, or alternatively, use the approach of Figure 11. This minimum value of β will depend upon the number (N) of beams in the bridge. It will be useful to estimate it for two or three values of N.

(a) $N = 10$

For one beam to fail, the smallest value of β will be found from $(1 - b)^{10} = 1 - 0.001$, where b is the portion of the probability distribution that corresponds to β . From this, $b = 0.0001$ closely. Expanding 26 with $\beta = \alpha + \epsilon$, we have, for small ϵ :

$$\begin{aligned} \text{fraction } b &= \frac{\epsilon^2 \cos 2\alpha - \frac{2}{3} \epsilon^3 \sin 2\alpha}{\cos 2\alpha - \left(\frac{\pi}{2} - 2\alpha \right) \sin 2\alpha} \\ \text{and for Red Pine, with } \alpha &= 28^\circ, \text{ we have } b = 8.287\epsilon^2 - 8.194\epsilon^3 \end{aligned} \quad (28)$$

Substituting $b = 0.0001$ in Equation 28 we readily find that $\epsilon = 0.0035$ radians = 0.20 degrees. Then there is a 1 in 1000 chance that a beam will fail in the interval $\theta = 28^\circ$ to $\theta = 28.2^\circ$, i.e. with a value of x between 0.5317 and 0.5362. We thus consider a situation in which (referring to equation 26) $x = 0.5317$ with $g(x) = 1.0$ and $x = 0.5362$ with $g(x) = 0.9$. The larger of these gives:

$$k = 0.532 \quad (29)$$

(b) $N = 20$

Proceeding similarly, $(1 - b)^{20} + 20b(1 - b)^{19} = 1 - 0.001$, whence from Equation 28, $b = 0.0024$ and $\epsilon = 0.0172$ radians = 1.0 degrees. We thus consider:

$$\begin{aligned} x &= \tan 28^\circ = 0.5317 \text{ with } g(x) = 1.0 \\ x &= \tan 29^\circ = 0.5543 \text{ with } g(x) = 0.90 \end{aligned}$$

The larger of these is again the first, so:

$$k = 0.532 \quad (30)$$

(c) $N = 30$

Similarly, $(1 - b)^{30} + 30b(1 - b)^{29} + 435b^2(1 - b)^{28} = 1 - 0.001$, whence: $b = 0.0066$ and $\epsilon = 0.0285$ radians = 1.63 degrees. We thus consider:

$$\begin{aligned} x &= \tan 28^\circ = 0.5317 \text{ with } g(x) = 1.00 \\ x &= \tan 29.63^\circ = 0.5188 \text{ with } g(x) = 0.90 \end{aligned}$$

The larger of these is again the first, so:

$$k = 0.532 \quad (31)$$

The lesson to be learned from the above work is that, at the 99.9% confidence level, there is enough "crowding" at the bottom end of the range of x that the failure of this first beam will precipitate collapse. Thus, the design value to take for k is the bottom end of the probability

distribution of $\left\{ \frac{(MOR)}{E_L} \right\} \left\{ \frac{(MOR)_{mean}}{(E_L)_{mean}} \right\}$, excluding only isolated values.

REFERENCES

- [1] Bakht, B. and Jaeger, L.G. "Computer Simulation of Failure in Timber Bridges" Proceedings of the Second International Conference on Computer Methods in Civil Engineering, Hangzhou, China, June 5-9, 1985.
- [2] Csagoly, P.F. and Taylor, R.J. "A Structural Wood System for Highway Bridges" IABSE Periodical 4/1980, pp 157-183.

NOTATION

E_L	Modulus of elasticity of a timber beam in the longitudinal direction
$(E_L)_{\text{mean}}$	Mean value of E_L for the species of timber concerned
MOR	Modulus of rupture of a timber beam
$(\text{MOR})_{\text{mean}}$	Mean value of MOR for the species of timber concerned
$\left\{ \frac{E_L}{(E_L)_{\text{mean}}} \right\}, \left\{ \frac{\text{MOR}}{(\text{MOR})_{\text{mean}}} \right\}$	Co-ordinate axes for representation of probability distribution of a given species of timber
(r, θ)	Polar co-ordinates in the plane of $\frac{E_L}{(E_L)_{\text{mean}}}$ and $\frac{\text{MOR}}{(\text{MOR})_{\text{mean}}}$, with θ being measured from the $\frac{E_L}{(E_L)_{\text{mean}}}$ axis
$p(r, \theta)$	A probability distribution
α	minimum value of θ for which the probability function $p(r, \theta)$ has non-zero values
$f(r), \phi(\theta)$	Variations of $p(r, \theta)$ in the r and θ direction respectively

β The value of θ corresponding to the maximum deflection in the cross-section of a bridge; i.e., the maximum value of γ .

γ The value θ corresponding to the deflection of a representative point on the cross-section of a bridge.

APPENDIX A/ VERIFICATION OF THE REDUCTION
FACTOR FOR THE CASE OF UNIFORM
DEFLECTION

It is of some interest from a theoretical as well as a practical standpoint to use the actual seventy pairings of MOR and E_L as given in Figure 1. We consider a bridge having seventy girders, one each from this grouping, and examine the progressive failure of beams as deflection increases. The first few values of $x \left\{ \frac{(MOR)}{E_L} / \frac{(MOR)_{mean}}{(E_L)_{mean}} \right\}$ are as follows:

0.433, 0.562, 0.566, 0.584, 0.594, 0.617, 0.622, 0.655, 0.677, 0.694,
0.700. If we multiply these by $\frac{70}{70}$, $\frac{69}{70}$, $\frac{68}{70}$... we get the value of k

just prior to the next breakage. The result is: 0.433, 0.554, 0.550,
0.559, 0.560, 0.573, 0.569, 0.590, 0.600, 0.605, 0.600, 0.590, and the
reduction factor is found to be 0.605, with nine beams already broken and
the tenth one about to fail. Note that failure of the second beam
immediately triggers failure of the third, but that k begins again to
rise thereafter, until the true maximum value of 0.605 is obtained.

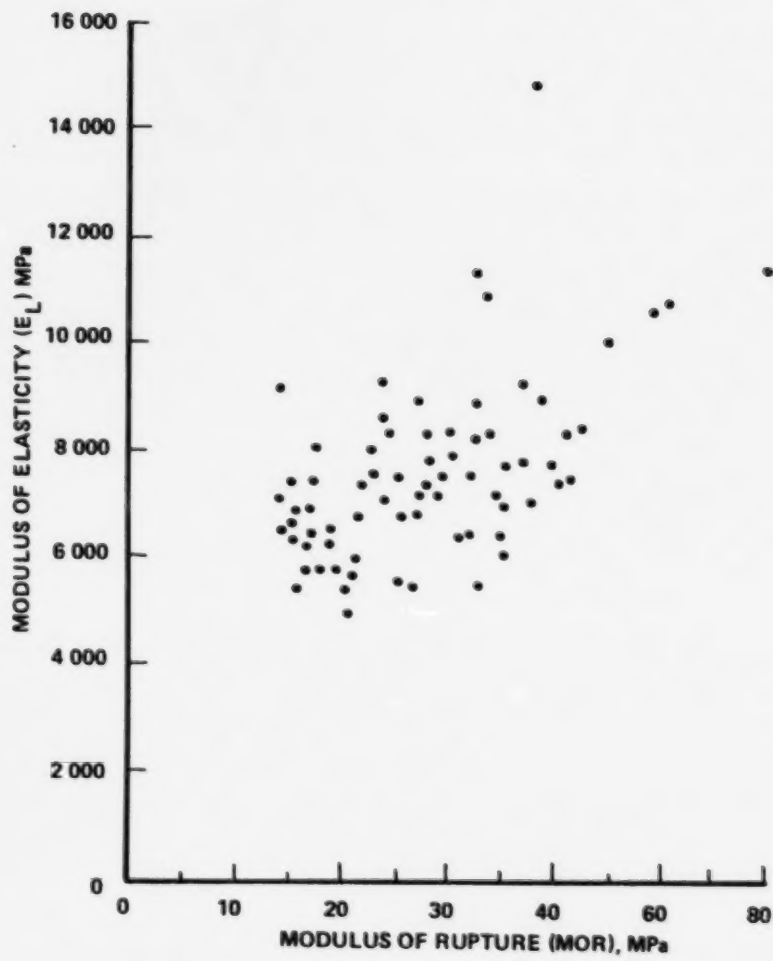


Figure 1/ Test Data on 70 Red Pine Specimens

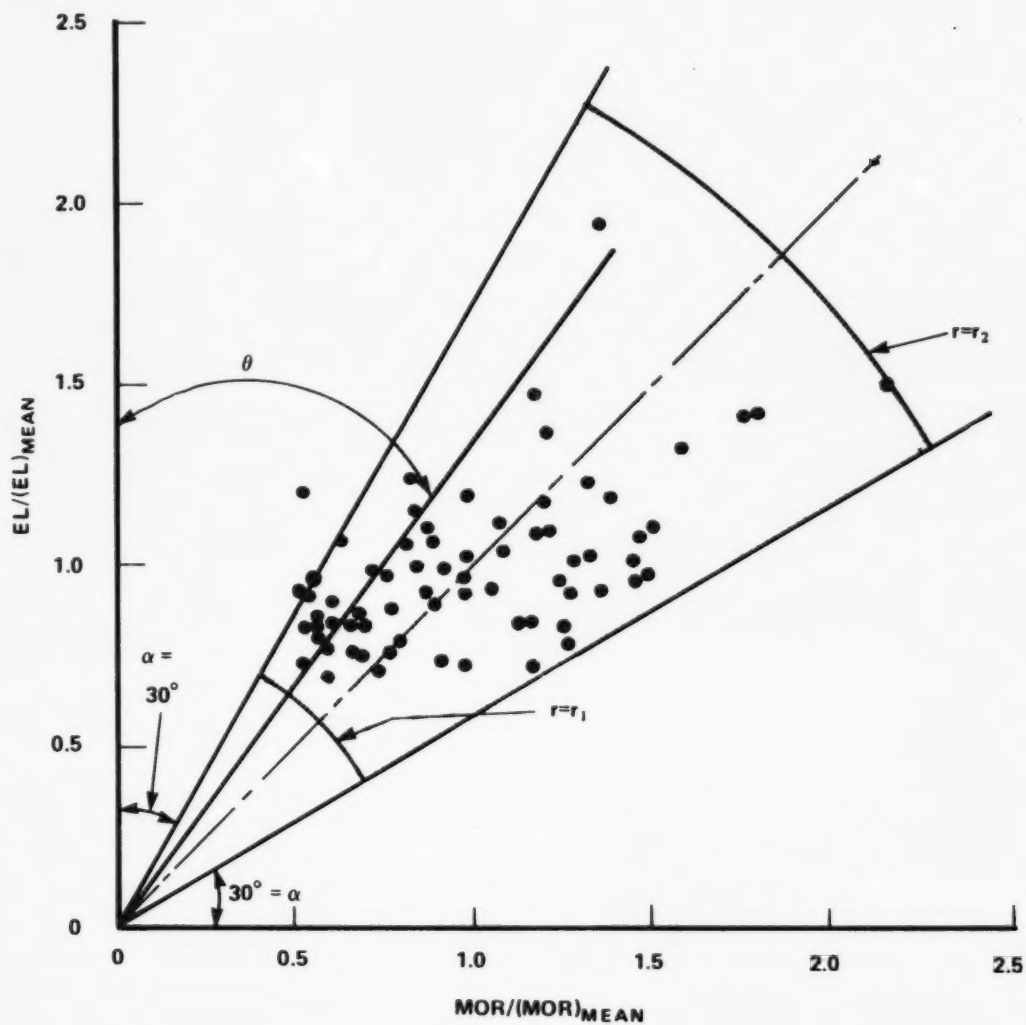


Figure 2/ Data on Red Pine Wood

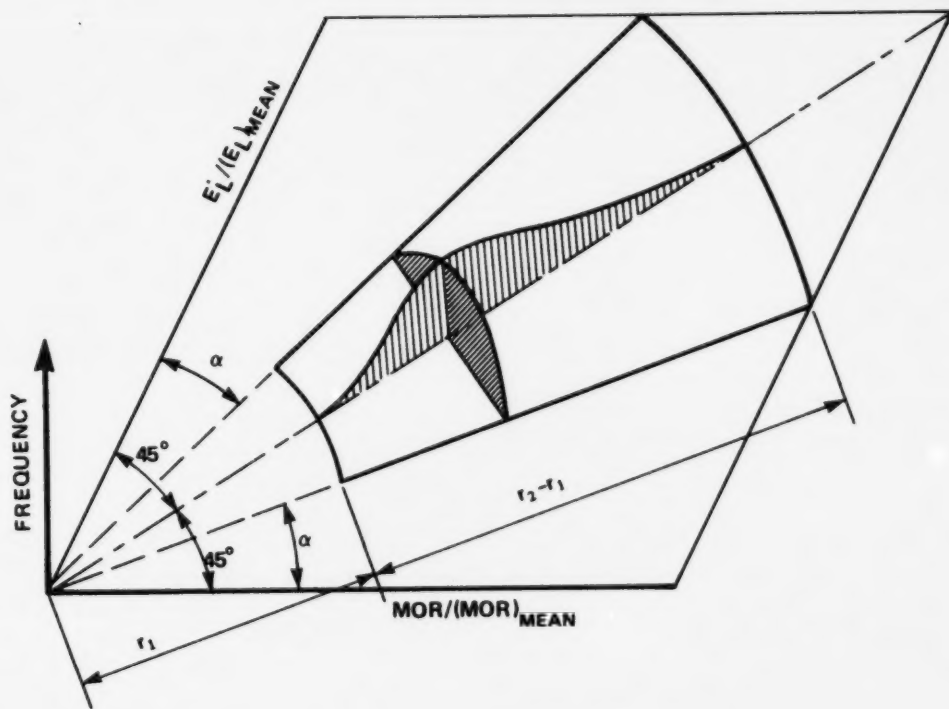


Figure 3(a)/ The Probability Function $p(r, \theta)$

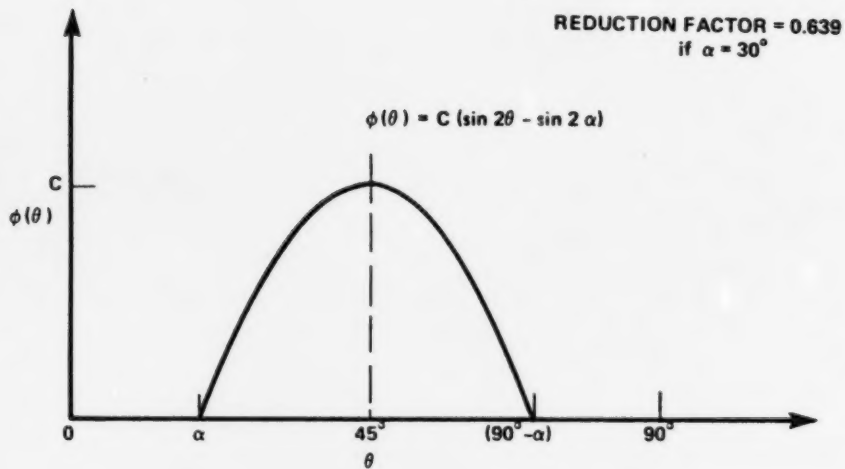


Figure 3(b)/ Plot of $\phi(\theta)$ Against θ

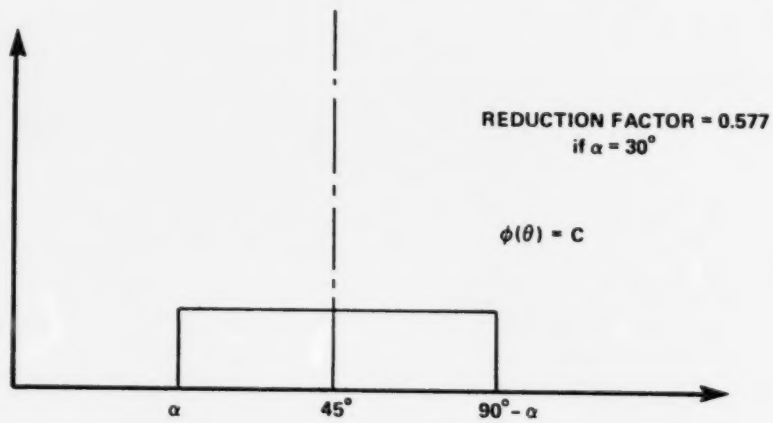


Figure 3(c)/ An Extreme Assumption for $\phi(\theta)$

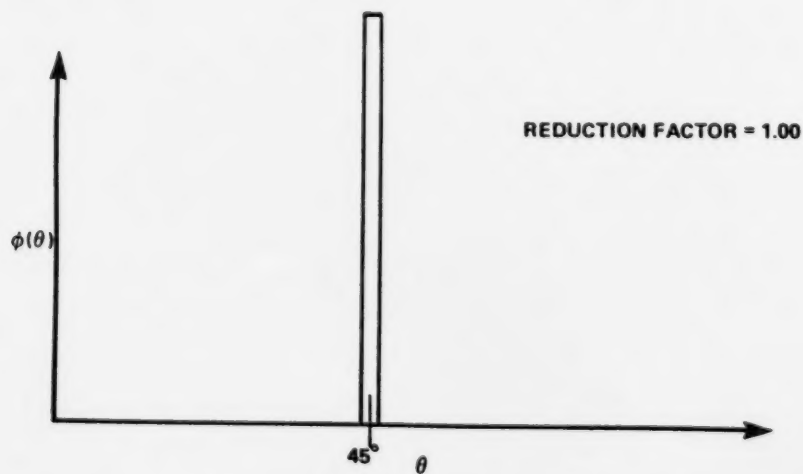


Figure 3(d)/ The Plot of $\phi(\theta)$ for Perfect Correlation of E_L and MOR
an Indefinitely High Spike of Vanishing width (Dirac Delta Function)

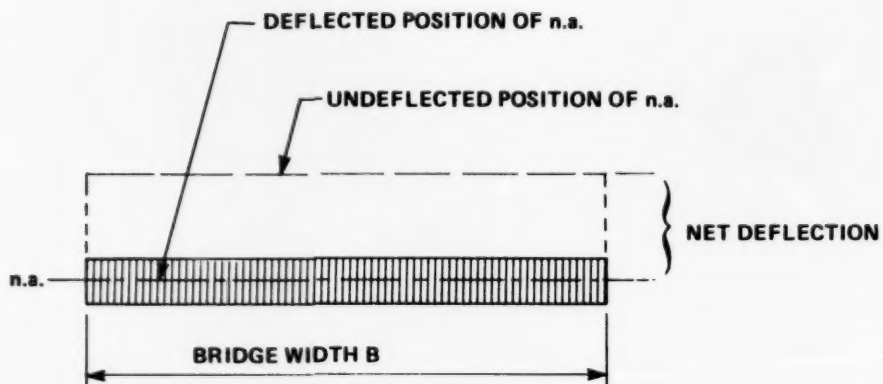


Figure 4/ Uniform Deflection Across a Bridge Cross-section

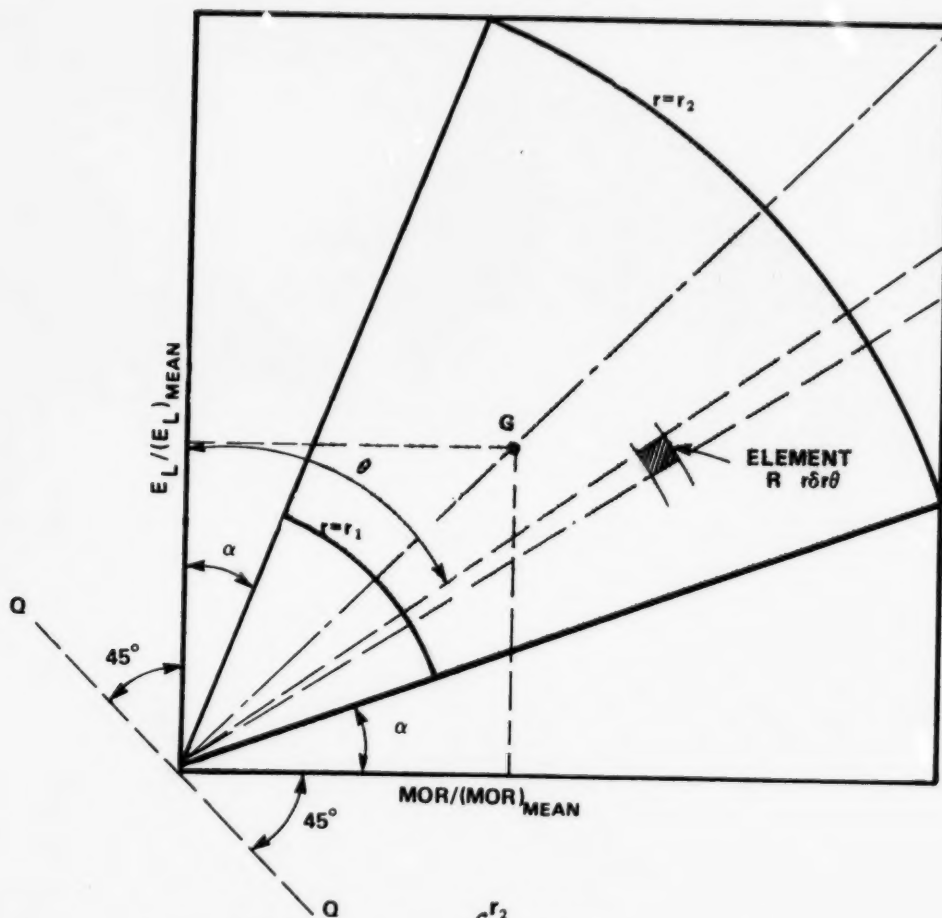


Figure 5(b)/ Evaluation of $\int_{r_1}^{r_2} f(r)r^2 dr$

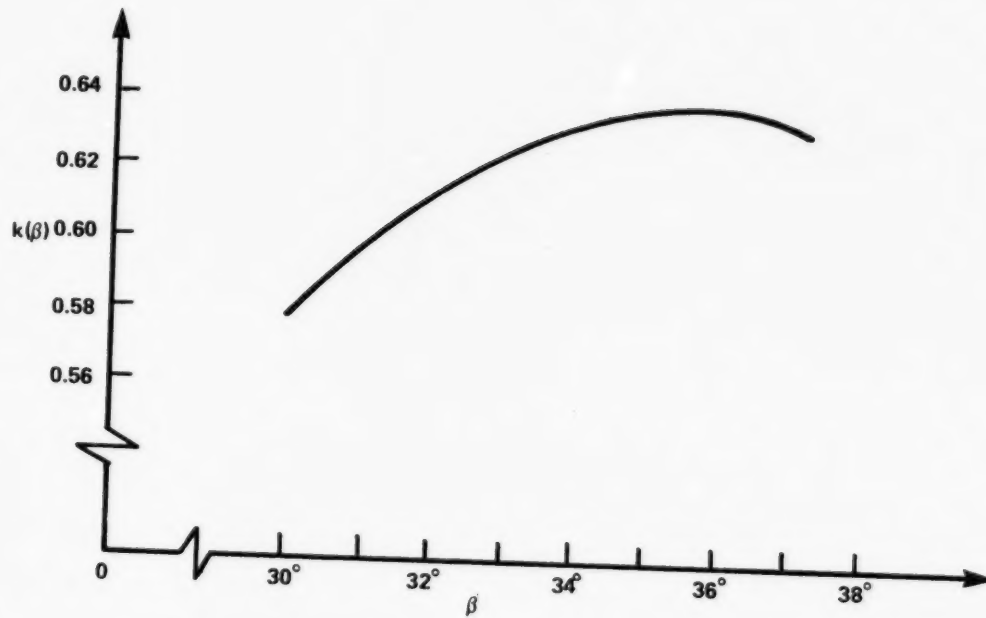


Figure 6/ Plot of $k(\beta)$ Against β

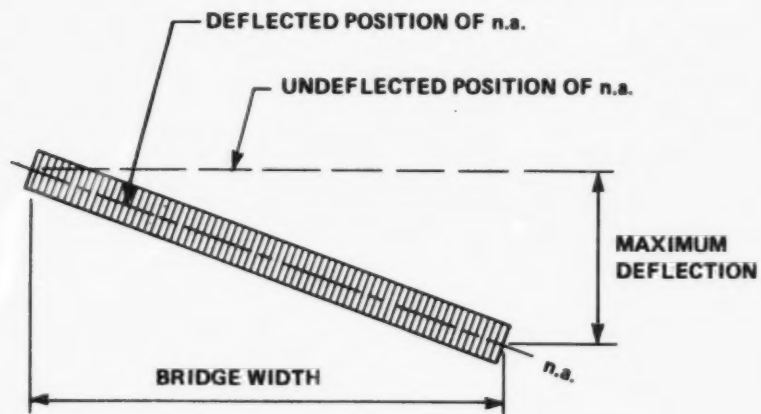
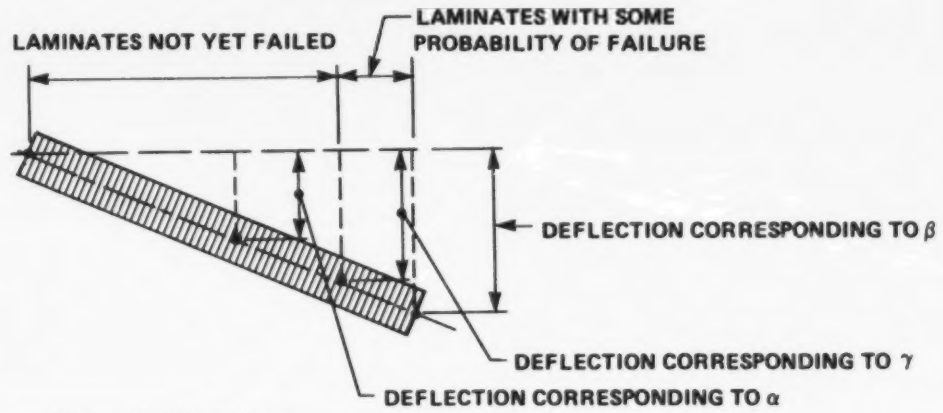
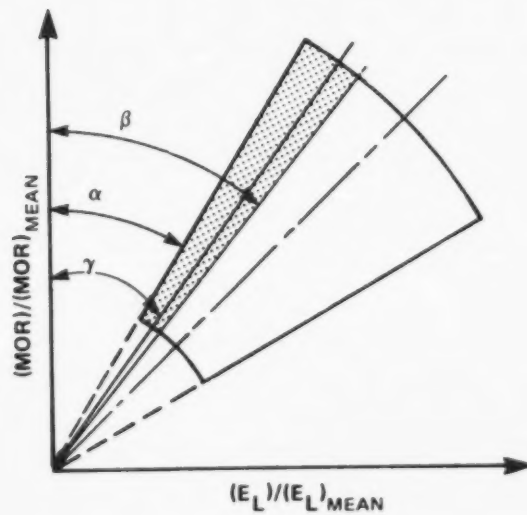


Figure 7/ Linear Deflection Across a Bridge Cross-section



a. DEFLECTION PROFILE



b.

Figure 8/ Identification of Failure Zone

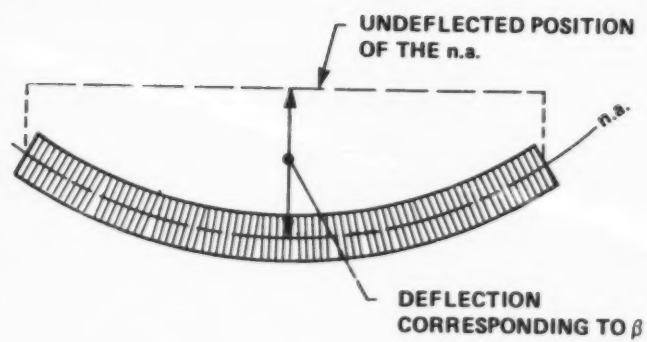
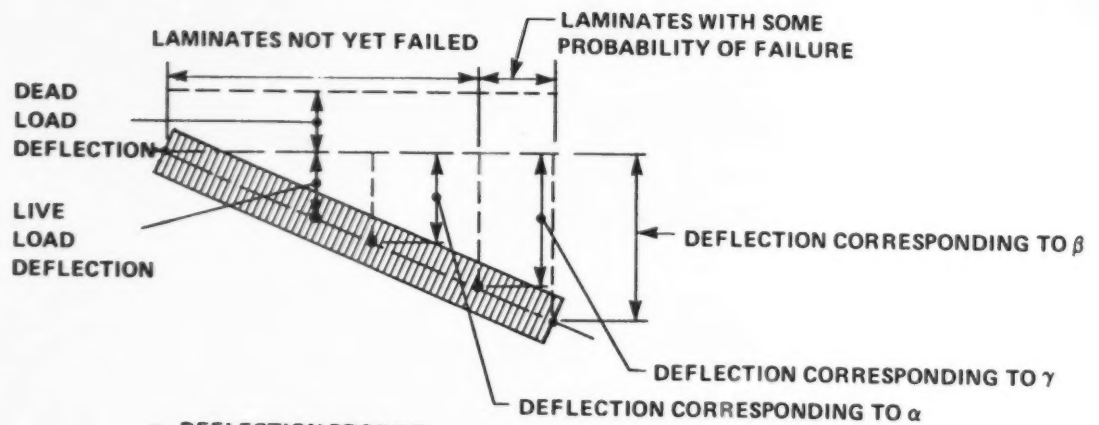
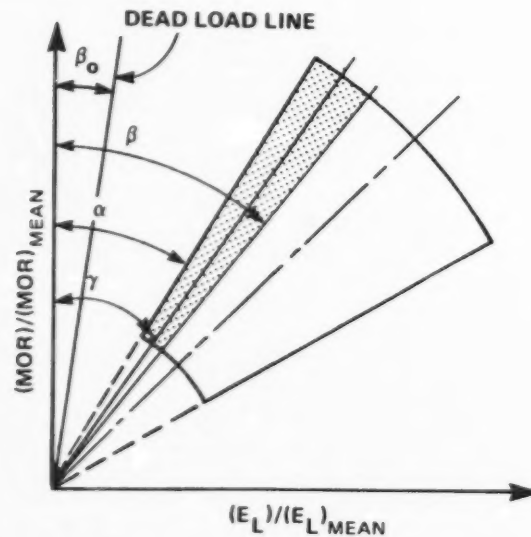


Figure 9/ A Non-uniform Deflection Profile



a. DEFLECTION PROFILE



b.

Figure 10/ Identification of Failure Zone by Considering Both Dead and Live Loads

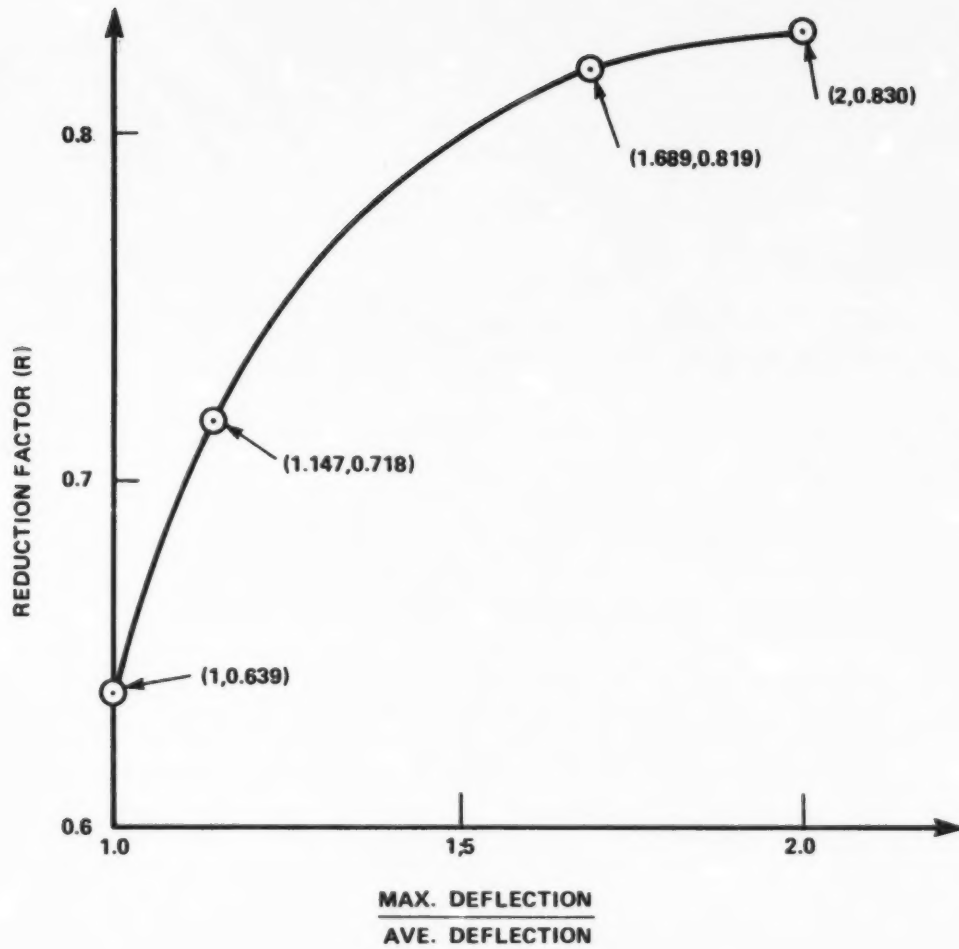


Figure 11/ Graph of Reduction Factor Against Deflection Ratio

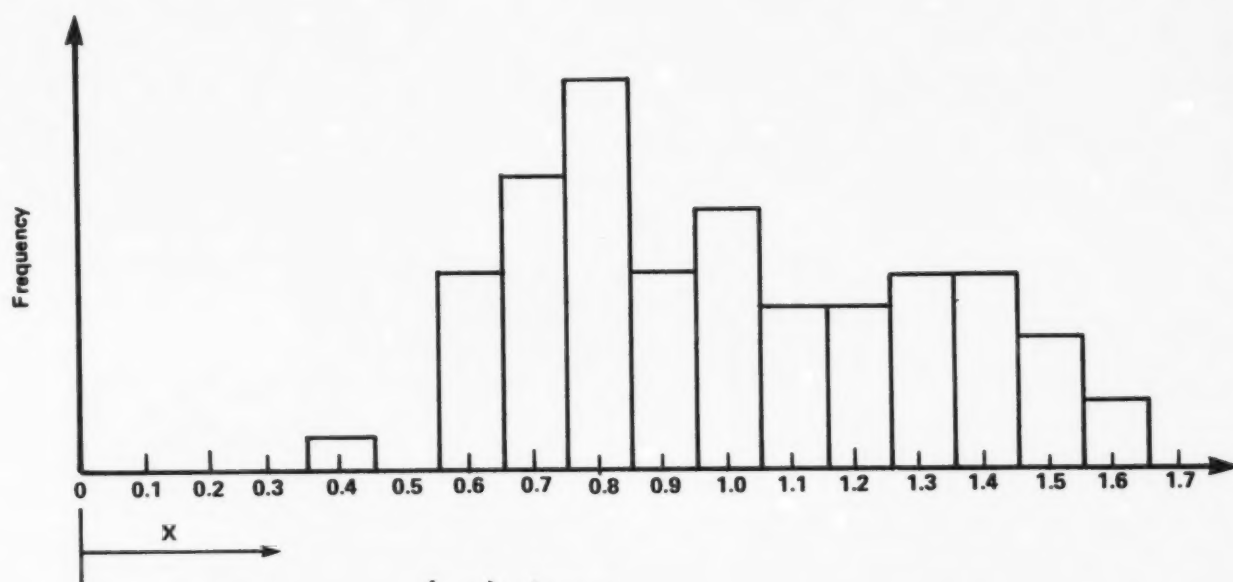


Figure 12(a)/ Histogram of $\left(\frac{MOR}{E_L} \right) \bigg/ \frac{(MOR)_{mean}}{(E_L)_{mean}}$ (Frequency Distribution)

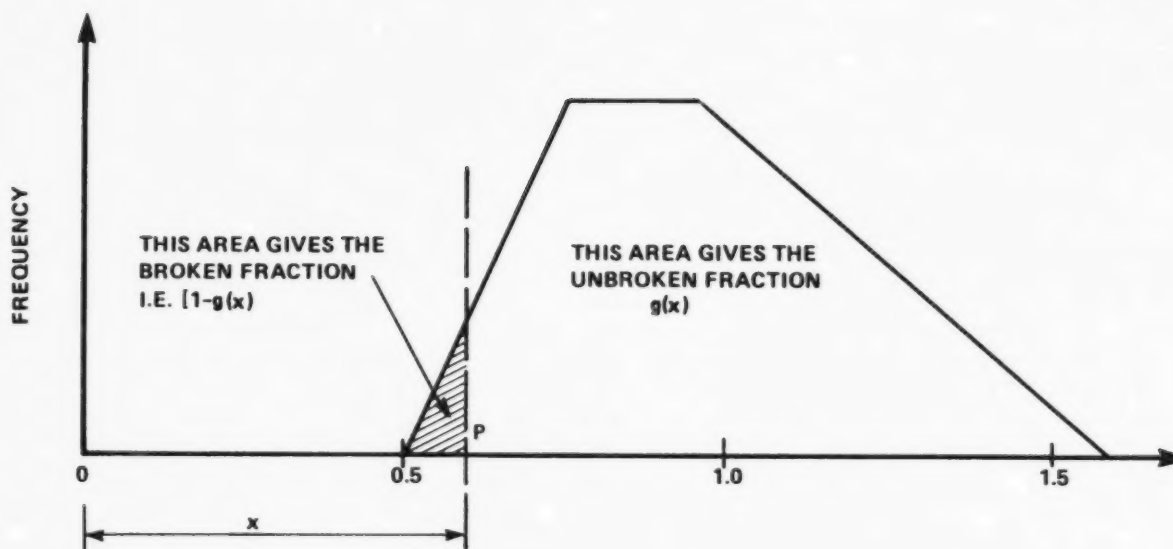


Figure 12(b)/ Simplified Frequency Distribution

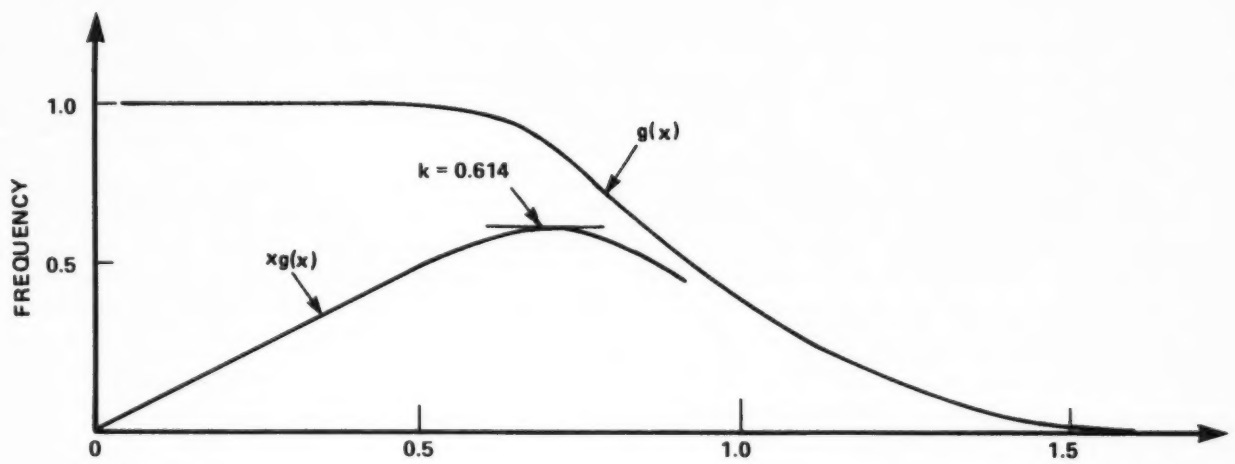


Figure 13/ Graphs of $g(x)$ and $xg(x)$